Error propagation in space-time prisms

Harvey J. Miller and **Tetsuo Kobayashi**
Department of Geography
University of Utah
Salt Lake City, Utah, USA
{harvey.miller, tetsuo.kobayashi}@geog.utah.edu

Walied Othman
Theoretical Computer Science Group
Hasselt University
Diepenbeek, Belgium
wailed.othman@UHasslet.be

Abstract— **The space-time prism demarcates all locations in space that a mobile object or person can occupy during an episode of potential or unobserved movement. The prism is a central concept in time geography as a measure of accessibility, and in mobile object databases as a measure of object location possibilities given sampling error. This paper develops an analytical approach to assessing error propagation in space-time prisms and prism-prism intersections. We analyze the geometry of the prisms to derive a core set of geometric problems involving the intersection of circles and ellipses. Analytical error propagation techniques such as the Taylor linearization method based on the first-order partial derivatives are not available since explicit functions describing the intersections and their derivatives are unwieldy. However, since we have implicit functions describing the core geometry, we modify this approach using an implicit function theorem that provides the required first-order partials without the explicit expressions.**

Keywords: mobile objects, space-time prism, error propagation, implicit function theorem

I. INTRODUCTION

The *space-time prism* is a central concept in time geography and in mobile objects databases. The prism demarcates all locations in space that a mobile object or person can occupy during a specific time interval. Determining the prism are measured parameters such as the locations of the object at the beginning and end of the interval, the times when these locations are occupied, the object's maximum travel velocity, and the minimum time when the object may be stationary during that interval. The prism is a central concept in time geography as a measure of accessibility of a mobile entity (person, animal) to the environment as well as the potential for contact with another mobile entity (in the case of prism-prism intersections; Hägerstrand 1970). More recently, the prism has been discovered in mobile object databases (MOD) as a measure of location possibilities given sampling error when tracking continuous movement only at discrete and finite moments in time (Kuijpers and Othman 2009). Although there is research on analytical and computational methods for calculating prisms in planar and network spaces, there is little work on propagation of measurement error from prism parameters.

This paper develops an analytical approach to error propagation in space-time prisms. We analyze the geometry of the prism and prism-prism intersection to derive a core set of geometric problems involving the intersection of circles and ellipses. This reduces the problem to determining the intersections and their error regions for combinations of these simple objects. Analytical error propagation techniques such as the Taylor linearization method based on first-order partial derivatives are not available in all cases since the explicit functions describing these intersections are unwieldy. However, since we have implicit functions describing the circle and ellipses, we modify this approach using an implicit function theorem that provides the required first-order partial derivatives without the need for explicit expressions.

II. BACKGROUND

A. Space-time prism

The classic space-time prism is a measure of movement possibilities in space with respect to time given constraints such as the need to be in designated places at specific times (the prism *anchors*), the available *time budget* for travel and activity participation, and the maximum travel velocity allowed in the environment (Hägerstrand 1970). It is also useful for demarcating the locations in space with respect to time when a tracked object may be present during periods between location samples (Kuijpers and Othman 2009). Fig. 1 illustrates a conceptual space-time prism.

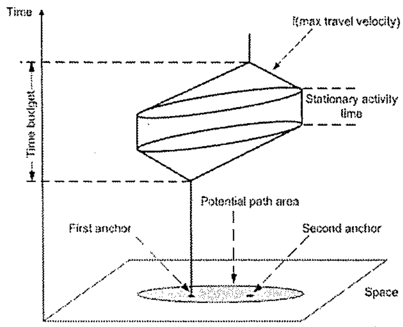

Figure 1. A space-time prism

Analytically, a space-time prism is determined by the parameters $(\mathbf{x}_i, \mathbf{x}_j, t_i, t_j, v_{ij}, a_{ij})$, defined as follows: $\mathbf{x}_i, \mathbf{x}_j$ are the first and second anchor locations with associated times t_i and t_j, v_{ij} is the maximum travel velocity and a_{ij} is any stationary activity time. At any moment during its existence, the prism is the intersection of up to three simple sets: i) the *future disc* $f_i(\mathbf{x}_i, t_i, t) \rightarrow \{\mathbf{x}, t\}$ capturing all locations that can be reached from the first anchor by the time $t_i + t \in (t_i, t_j)$; ii) the *past disc*

$p_j(\mathbf{x}_j, t_j, t) \rightarrow \{\mathbf{x}, t\}$ encompassing all locations that can reach the second anchor in the time remaining $t_j - t \in (t_i, t_j)$, and; iii) the *geo-ellipse* $g_{ij}(x_i, x_j, t_i, t_j, a_{ij}, t) \rightarrow \{\mathbf{x}\}$ that constrains the prism locations to account for any stationary activity time a_{ij} during the time interval. The latter set is equivalent to the *potential path area* of classical time geography. The boundaries of these sets are (Miller 2005):

$$f_i: \quad \|\mathbf{x} - \mathbf{x}_i\| - (t - t_i)v_{ij} = 0 \qquad (1)$$

$$p_j: \quad \|\mathbf{x}_j - \mathbf{x}\| - (t_j - t)v_{ij} = 0 \qquad (2)$$

$$g_{ij}: \quad \|\mathbf{x} - \mathbf{x}_i\| + \|\mathbf{x}_j - \mathbf{x}\| - (t_j - t_i - a_{ij})v_{ij} = 0 \qquad (3)$$

In two-dimensional planar space, these sets are circles and ellipses.

Solving for the prism at any time $t \in (t_i, t_j)$ never requires finding the intersection for all three of the sets. During much of the prism existence, one or two of the spatial sets totally encompass one or two of the other sets, meaning that the latter can be ignored. This can be extended to prism-prism intersections, with the worse-case requiring solving for the intersection of two circles and two ellipses in two-dimensional planar space.

B. *Error propagation methods*

The prism parameters $(\mathbf{x}_i, \mathbf{x}_j, t_i, t_j, v_{ij}, a_{ij})$ often contain error, and therefore we are interested in estimating the propagation of this error to the intersections of the sets described in equations (1)-(3). A well-known analytical error method is the Taylor linearization technique. The Taylor method estimates error propagation based on the first-order partial derivatives of the propagating function (see Heuvelink 1998). Leung, Ma and Goodchild (2004) provide a canonical statement of this approach as applied to spatial error propagation.

To apply analytical error propagation methods to the prism and prism-prism intersections, we need explicit functions describing the intersections of the boundaries in equations (1) - (3) as well as the Jacobian matrix of their first-order partial derivatives. However, despite the elegant geometry of circles and ellipses, explicit functions describing these intersection locations and their derivatives are unwieldy in many cases; requiring solving for the roots of higher order polynomials. However, there is an alternative strategy that exploits equations (1) - (3) are implicit functions (that is, stated in the form $f(x,y) = 0$ instead of explicitly as $f(x) = y$).

C. *Implicit function theorm*

The *implicit function theorem* (IFT) states that, if the partial derivatives of an implicit function satisfy some mild conditions in a local region, an explicit differentiable function also exists in that region. Although this function is unknown, its Jacobian matrix is available as a corollary of the theorem. This Jacobian matrix provides the first order partials required for error propagation (Rudin 1976).

Implicit function theorem (IFT). Let S be an open subset of $\Re^{k+p}$ with elements $(y_1, \ldots, y_k, x_1, \ldots x_p) = (\mathbf{y}, \mathbf{x})$ where $\mathbf{y} = (y_1, \ldots, y_k)$, $\mathbf{x} = (x_1, \ldots, x_p)$ and $g_i : \Re^{k+p} \rightarrow \Re$ satisfies $g_i(\mathbf{y}_0, \mathbf{x}_0) = 0$ where $i = 1, \ldots, p$ at some point $(\mathbf{x}_0, \mathbf{y}_0) \in S$. If all $\partial g_i / \partial x_i$ are continuous and $\det\left(\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\Bigg|_{(\mathbf{y}_0, \mathbf{x}_0)}\right) \neq 0$ then there exists open regions $W \subseteq \Re^p$ of $\mathbf{x}_0$ and $V \subseteq \Re^k$ of $\mathbf{y}_0$ such that $W \times V \subseteq S$ and there exists functions f_i such that $f: V \rightarrow W : \mathbf{y} \rightarrow f(\mathbf{y}) = (f_1(\mathbf{y}), \ldots, f_p(\mathbf{y})) = (x_1, \ldots x_p) = \mathbf{x}$. ■

The functions f are continuous and $\left(\frac{\partial f_i}{\partial y_j}\right)_{ij} = -\left(\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}\right)^{-1}\left(\left(\frac{\partial g_i}{\partial y_j}\right)_{ij}\right)$ where the left hand side is evaluated in $\mathbf{y}$ and the right hand side in $(\mathbf{y}, \mathbf{x}) = (\mathbf{y}, f(\mathbf{y}))$. This matrix allows us to calculate the required Jacobian matrix for the explicit function without having to compute the function itself. Benichou and Gail (1989) use this theorem to generalize the delta method for estimating the probability distributions for functions of normal random variables; a method that also exploits the first-order Taylor method.

III. ANALYTICAL METHODS FOR ERROR PROPAGATION IN SPACE-TIME PRISMS

The general strategy is as follows. For a given space-time prism or prism-prism intersection at a moment in time, we determine the equivalent intersection problem involving circles and/or ellipses in planar two-dimensional space. Given this intersection problem, we check whether the required Jacobian matrix can be calculated directly (as in the Taylor method); if not, we must check whether the IFT conditions are satisfied. If neither set of conditions are satisfied, then we must resort to an approximation.

Regardless of whether we use the Taylor or IFT strategy, we need a one-to-one relationship between the parameters that have measurement error and the variables for which we wish to estimate the error. Under some geometric conditions, this one-to-one mapping is possible when considering the circle or ellipse as embedded within a cone or cylinder; this allows derivation of a spatio-temporal error ellipsoid. Under other conditions, a one-to-one relationship is only possible if treat the circles and ellipses as static; in these cases we can only derive a spatial error band.

A. *Overview of intersection problems and error solutions*

Tables 1, 2 and 3 summarize the possible geometric intersection problems that can occur in prism and prism-prism intersections, the corresponding error propagation solution strategy, and whether the solution is a spatial error band or a spatio-temporal ellipsoid. We classify these cases by examining the rank of the corresponding Jacobian matrix.

In some cases the Taylor or IFT methods are sufficient. In some cases, however, only an approximation or partial view of the error is possible. In other cases, the Taylor or IFT strategies work but only for a subset of the objects, meaning that we can calculate only a partial view of the error. The case of two ellipses touching over an interval is open: we do not have a solution.

TABLE I. PRISM INTERSECTION PROBLEMS: TWO OBJECTS

Case	Solution strategy	Solution type
Two circles		
Proper intersection	IFT	Spatial
Touching over an interval	Taylor	Spatio-temporal
Touching for a moment	Taylor	Spatio-temporal
Circle-ellipse		
Proper intersection	IFT (approximation based on discrete temporal intervals)	Spatio-temporal
Touching at a moment	IFT	Spatio-temporal
Two ellipses		
Proper intersection	IFT	Spatial
Touching over an interval	Open	

TABLE II. PRISM INTERSECTION PROBLEMS: THREE OBJECTS

Case	Solution strategy	Solution type
Three circles		
Proper intersection	IFT (most cases)	Spatial
Two touch at a moment	IFT	Spatio-temporal
Two touch over an interval	Taylor (partial – two objects)	Spatial
Three touch at a moment	IFT (partial – two objects)	Spatio-temporal
Three touch for a moment, but two touch over an interval	Taylor (partial – two objects)	Spatial
Three touch over an interval	Taylor (partial – two objects)	Spatio-temporal
Two circles and an ellipse		
Proper intersection	IFT (most cases)	Spatial
One circle and ellipse touch for a moment, second circle does not	IFT	Spatio-temporal
Two circles and ellipse touch (circles for a moment or an interval)	IFT (partial – circles touching ellipse) Taylor (partial – circles touching each other)	Spatial, spatio-temporal
Two circles touch (for a moment or an interval) but the ellipse does not	Taylor (partial – touching circles) IFT (partial – all other combinations of two objects)	Spatial
Ellipse with circles centered on each foci, one shrinking and the other growing at the same rate	IFT (partial – two objects)	Spatio-temporal
A circle and two ellipses		
Proper intersection	IFT	Spatio-temporal
Circle and ellipse touch, second ellipse does not touch	IFT (partial – two objects)	Spatio-temporal
Two ellipses touch, circle touches or not	IFT (partial – circle and ellipse)	Spatial, spatio-temporal

TABLE III. PRISM INTERSECTION PROBELMS: FOUR OBJECTS

Case	Solution strategy	Solution type
Intersecting at a moment	Taylor and IFT (partial - three objects)	Spatial, spatio-temporal
Intersecting over an interval	Taylor (partial – two objects)	Spatial, spatio-temporal

Details of the solution strategies can be found in Kobayashi, Miller and Othman (unpub). We will now provide an illustrative example: the case of two circles forming a proper intersection in two dimensional planar space.

B. Example calculation: Two circles in 2D planar space

Assume we have two circles, given by the equations:

$$\begin{cases} g_1 \equiv (x-x_1)^2 + (y-y_1)^2 - (t-t_1)^2 v_1^2 = 0 \\ g_2 \equiv (x-x_2)^2 + (y-y_2)^2 - (t-t_2)^2 v_2^2 = 0 \end{cases} \tag{4}$$

The measured parameters are $x_1, y_1, t_1, v_1, x_2 y_2, t_2$ and v_2. We have two constraints and three variables, which is a one-dimensional set. In this case the Taylor method does not work either because a function f that maps $(x_1, y_1, t_1, v_1, x_2 y_2, t_2, v_2)$ to a single (x, y, t) does not exist. Also, the number of implicit functions does not equal the number of variables we need to estimate, hence the IFT method apparently does not work either.

To overcome this difficulty we reduce the number of variables for which we want to provide estimates. We treat time as a parameter and not as a variable for which we want to estimate the error. We therefore have:

$$\begin{cases} g_1 = (x-x_1)^2 + (y-y_1)^2 - (t-t_1)^2 v_1^2 \\ g_2 = (x-x_2)^2 + (y-y_2)^2 - (t-t_2)^2 v_2^2 \end{cases} \tag{5}$$

$$\left(\left(\frac{\partial g_i}{\partial \mathbf{x}_j} \right)_{ij} \right) = \begin{pmatrix} 2(x-x_1) & 2(y-y_1) \\ 2(x-x_2) & 2(y-y_2) \end{pmatrix} \tag{6}$$

The determinant of the matrix above will be zero if and only if (x, y) is collinear with (x_1, y_1) and (x_2, y_2): this occurs if and only if the circles touch. In all other cases, the IFT conditions are satisfied.

We require the matrices $\mathbf{J}$ and $\mathbf{H}$ where:

$$\mathrm{H} = \begin{pmatrix} -2(x-x_1) & -2(y-y_1) & -2v_1^2(t-t_1) & -2v_1(t-t_1)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2(x-x_2) & -2(y-y_2) & -2v_2^2(t-t_2) & -2v_2(t-t_2)^2 \end{pmatrix} \tag{7}$$

and $\mathbf{J}$ provided by equation **(6)**. Let Σ be the 8×8 covariance matrix of the measured variables and let $\mathbf{K} = -\mathbf{J}^{-1}\mathbf{H}$. The covariance matrix of (x, y) is $\mathbf{K}\Sigma\mathbf{K}^{\mathrm{T}}$, where $\mathbf{K}$ is the Jacobian matrix of the (unknown) function that maps $(x_1, y_1, t_1, v_1, x_2, y_2, t_2, v_2)$ to (x, y). The variables, at each moment in time where the proper intersection occurs, are random variables under a bivariate normal distribution with covariance matrix $\mathbf{K}\Sigma\mathbf{K}^{\mathrm{T}}$. The expression $(X-x,Y-y)\mathbf{K}\,\Sigma\,\mathbf{K}^{\mathrm{T}}(X-x,Y-y)^{\mathrm{T}}$ corresponds to a chi square distribution for the variables (X,Y) with 2 degrees of freedom. A 95% confidence interval for the variables (X,Y) is obtained by the expression:

$$(X-x,Y-y)\mathbf{K}\,\Sigma\,\mathbf{K}^{\mathrm{T}}(X-x,Y-y)^{\mathrm{T}} \le 5.991 \qquad (8)$$

We implemented this method in Mathematica; Figure 2 illustrates the intersection problem of two growing circles in time (left side) and the 95% confidence interval (right side). The error band degenerates to a line segment as the intersection approaches the moment where the two circles touch (towards the bottom half of the intersection in Figure 2; this corresponds to the confidence bands degenerating to a horizontal band in the right hand side of the figure). This is because **J** approaches a singular matrix and can not be inverted at that moment.

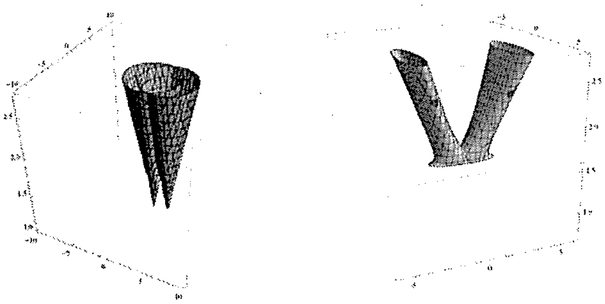

Figure 2. Two growing circles and the 95% confidence regions arouund their intersection

IV. Conclusion

The space-time prism is a central concept in time geography as a measure of accessibility and interaction possibilities, as well as mobile object databases and related applications as a measure of uncertainty related to sampling error. This paper develops analytical methods for assessing error in space-time prisms and their intersections. Geometric analysis reduces the problem to determining the error regions around the intersection points of simple objects (discs and ellipsoids in general; circles and ellipses in two dimensional space). Using the spatial error propagation law articulated by Leung, Ma and Goodchild (2004) as a framework, we supplement this approach with methods based on an implicit function theorem that provides the required first-order partial derivatives for the implicit functions describing the simple objects involved in calculating prisms and their intersections.

The analytical methods in this paper have some limitations. First, we do not have a complete treatment of error for the prism or prism-prism intersections: in some cases we are forced to use approximations and in other cases we can only describe error for a subset of the objects involved. One geometric intersection problem (two ellipses touching over an interval) is still open. Second, these methods for prism-prism intersections are limited to a small number of prisms. In this regard the analytical methods share a weakness with Monte Carlo simulation: they are not scalable. However, they are more generalizable than Monte Carlo methods. The methods in this paper can be used to develop approximations such as tight upper bounds on the true error regions, or even scalable numeric methods.

The methods in this paper are limited to classic (constant velocity) prisms. Also required are methods for the more general case of non-constant velocities. Anchor error in network time prisms have been addressed (Kuijpers et al. in press). Recently, Miller and Bridwell (2009) generalize the space-time prism to the case for isotropic and anisotropic velocity fields.

Another research frontier is the general applicability of the implicit function method for analyzing error propagation in spatial analysis. Analytical error propagation methods require explicit functions, and there is probably only a subset of spatial analytic operations that can be described by tractable explicit functions. We have demonstrated in this paper that the implicit function method allows extension of the error propagation methods to a class of spatial operations that involve the intersections of circle and ellipses. It is an open question to determine other geospatial operations that can be stated more tractably as implicit functions to extend spatial error propagation methods to these operations as well.

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