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# An optimal control method for high accuracy surface modeling and its application to DTM construction

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## Abstract

*An optimal control method of high accuracy surface modelling (HASM-OC) is developed to improve DTM as accurate as possible in this paper. One numerical test is conducted to validate HASM-OC by comparing with Thin Plate Spline (TPS). Datasets with various contour intervals are selected to simulate DTMs in different spatial resolutions. Then, retrieved contour lines, respectively by HASM-OC and TPS, are compared with the original ones. The results indicate that the contour lines retrieved by HASM-OC almost coincide with the original contour lines in appropriate spatial resolutions while the ones done by TPS have a considerable difference from the original ones. The difference between the retrieved contour lines by HASM-OC and the original ones is less than 6.5% of grid size. Errors of contour lines retrieved by HASM-OC monotonically decease with spatial resolution becoming finer but TPS presents oscillating errors. HASM-OC has much higher accuracy and performs much more stable.*

**Keywords**: HASM, TPS, optimal control, DTM, accuracy, error

## 1. Introduction

Digital terrain models (DTMs) are representations of the terrain elevation as a function of geographic location (Raaflaub and Collins, 2006). DTMs are typically represented in two formats: contour maps, where the surface is represented by lines of constant elevation at even intervals; or point heights, where the surface elevation is sampled on either regular or irregular grids. In this paper, the method of high accuracy surface modeling (HASM) (Yue *et al*., 2007, 2010) is combined with optimal control theory to make accuracy of DTM as high as possible.

Optimal control theory is to determine the control signals that cause a process to satisfy the physical constraints and at the same time minimize some performance criterion (Kirk, 1970). It is a common objective in many research fields to develop control function for commanding a dynamic system to a desired output or for augmenting the system's stability (Stengel, 1994). If the control objective can be formulated as a quantitative criterion, then optimization of this criterion establishes a feasible design structure for the control function. Optimal control concerns the properties of control functions that give solutions which minimize a measure of performance. The differential equation describes the dynamic response of the mechanism to be controlled, which depends on the control function (Vinter, 2000).

The necessary theoretical advances of optimal control theory have been made since late 1930s (Tan and Bennett, 1984). Message and noise mixture was filtered by using a linear operator to minimize the noise (Wiener, 1950). Development of the filtering theory solved many control problems. Dynamic programming and maximum principle provided the basis for the development of modern control theory. Dynamic programming was designed to treat multi-stage processes possessing certain invariant aspects, which allowed solution of multi-stage control problem from the final to the initial state by the N-stage return and reduced a single N-dimesional problem to a sequence of N one-dimensional problems (Bellman, 1954, 1957). Maximum principle could be deduced from the principle of dynamic programming (Pontraygin *et al*., 1962). Maximum principle was a set of necessary conditions for a control function to be optimal, which was appropriate for deterministic control problem. In the certainty equivalence principle, it was concluded that the optimal control settings of linear models with an additive random term can be found in the same way as deterministic models (Simon, 1956; Theil, 1957). It was a significant development that the control problem was recognized as the inverse of the estimation problem (Stolz, 1960). Nonsmooth analysis and viscosity method was an important breakthrough in the 1970s. It was demonstrated that much of optimization and analysis which had evolved under traditional smoothness assumptions could be developed in a general nonsmooth setting (Clarke, 1983).

Optimal control theory has been widely applied in many fields with development of computer science and technology. For instance, minimum-fuel rocket trajectories were calculated to achieve successful navigation towards reaching the Moon (Breakwell and Dixon, 1975). Thin plate spline (TPS) for spatial interpolation was developed by minimizing bending energy (Duchon, 1977). An optimal control model was used in greenhouse climate management during the cultivation of a lettuce crop (van Henten, 2003). Optimal control theory was employed to develop integrating simulation models for the operation and control of urban wastewater systems (Butler and Schutze, 2005). An algorithm for optimum control of the ground-water conditions in a drained mass for the purpose of formation of a uniform distribution of the ground-water level was derived for the condition that the rate of decrease in the ground-water level reach a maximum value at a minimum root-mean-square deviation of the ground-water level from a certain value for a definite period of time (Bobarykin and Latyshev, 2007). An optimal control function was derived to asymptotically stabilize chaotic rotations and minimize the required like-energy cost (El-Gohary, 2009). An optimal control model was used to determine the optimized energy losses in multi-boiler steam systems (Bujak, 2009).

The existence and uniqueness of the optimal controller were established by using Ekeland's principle to deal with an optimal control problem for a kind of age-dependent biological population systems (Chen and He, 2009). Optimal control was applied to optimal harvesting of renewable resources with alternative use (Piazza and Rapaport, 2009). An optimal control model was designed for the load shifting problem in energy management to improve energy efficiency through the control of conveyor belts (Middelberg *et al*., 2009). The optimal purification of a polluted section of a river was formulated as a hyperbolic optimal control problem with control constraints, where the state system was given by shallow water equations coupled with the pollutant concentration equation, the control was the flux of injected water, and the objective function was related to the total quantity of injected water and the pollution thresholds (Alvarez-Vazquez *et al*., 2009). However, measurement accuracy of the relevant variables and computational difficulties associated with determination of an optimal control often impeded the economical implementation of an optimal design (Athans and Falb, 2007).

In this paper, taking DTM as an example, an optimal control method of high accuracy surface modeling (HASM-OC) is developed and validated to meet the specific requirement for extremely high accuracy, although it has been demonstrated that methods of high accuracy surface modeling (HASM) has a much higher accuracy than classic methods (Yue *et al*., 2007, 2009).

## 2. HASM-OC

Suppose $\left\{\left(x_i, y_j\right)\middle| x_i = i \cdot h,\quad y_j = j \cdot h,\quad 0 \le i \le I+1,\quad 0 \le j \le J+1\right\}$ is an orthogonal division of computational domain $\Omega$ and $h$ is grid cell size of the division. Then, the iterative formulation of HASM can be formulated as (Yue *et al*., 2007, 2009),

$$\begin{cases} \dfrac{f_{\mathrm{i+1,j}}^{(n+1)} - 2f_{\mathrm{i,j}}^{(n+1)} + f_{\mathrm{i-1,j}}^{(n+1)}}{h^2} = \left(\Gamma_{11}^1\right)_{\mathrm{i,j}}^{(n)} \dfrac{f_{\mathrm{i+1,j}}^{(n)} - f_{\mathrm{i-1,j}}^{(n)}}{2h} + \left(\Gamma_{11}^2\right)_{\mathrm{i,j}}^{(n)} \dfrac{f_{\mathrm{i,j+1}}^{(n)} - f_{\mathrm{i,j-1}}^{(n)}}{2h} + \dfrac{L_{\mathrm{i,j}}^{(n)}}{\sqrt{E_{\mathrm{i,j}}^{(n)} + G_{\mathrm{i,j}}^{(n)} - 1}} \\ \dfrac{f_{\mathrm{i,j+1}}^{(n+1)} - 2f_{\mathrm{i,j}}^{(n+1)} + f_{\mathrm{i,j-1}}^{(n+1)}}{h^2} = \left(\Gamma_{22}^1\right)_{\mathrm{i,j}}^{(n)} \dfrac{f_{\mathrm{i+1,j}}^{(n)} - f_{\mathrm{i-1,j}}^{(n)}}{2h} + \left(\Gamma_{22}^2\right)_{\mathrm{i,j}}^{(n)} \dfrac{f_{\mathrm{i,j+1}}^{(n)} - f_{\mathrm{i,j-1}}^{(n)}}{2h} + \dfrac{N_{\mathrm{i,j}}^{(n)}}{\sqrt{E_{\mathrm{i,j}}^{(n)} + G_{\mathrm{i,j}}^{(n)} - 1}} \end{cases}$$

(1)

where

$$E_{\mathrm{i,j}}^{(n)} = 1 + \left(\frac{f_{\mathrm{i+1,j}}^{(n)} - f_{\mathrm{i-1,j}}^{(n)}}{2h}\right)^2;$$

$$F_{\mathrm{i,j}}^{(n)} = \left(\frac{f_{\mathrm{i+1,j}}^{(n)} - f_{\mathrm{i-1,j}}^{(n)}}{2h}\right)\left(\frac{f_{\mathrm{i,j+1}}^{(n)} - f_{\mathrm{i,j-1}}^{(n)}}{2h}\right);$$

$$G_{\mathrm{i,j}}^{(n)} = 1 + \left(\frac{f_{\mathrm{i,j+1}}^{(n)} - f_{\mathrm{i,j-1}}^{(n)}}{2h}\right)^2;$$

$$L_{\mathrm{i,j}}^{(n)}=\frac{f_{\mathrm{i+1,j}}^{(n)}-2f_{\mathrm{i,j}}^{(n)}+f_{\mathrm{i-1,j}}^{(n)}}{h^2\sqrt{1+\left(\frac{f_{\mathrm{i+1,j}}^{(n)}-f_{\mathrm{i-1,j}}^{(n)}}{2h}\right)^2+\left(\frac{f_{\mathrm{i,j+1}}^{(n)}-f_{\mathrm{i,j-1}}^{(n)}}{2h}\right)^2}};$$

$$N_{\mathrm{i,j}}^{(n)}=\frac{f_{\mathrm{i,j+1}}^{(n)}-2f_{\mathrm{i,j}}^{(n)}+f_{\mathrm{i,j-1}}^{(n)}}{h^2\sqrt{1+\left(\frac{f_{\mathrm{i+1,j}}^{(n)}-f_{\mathrm{i-1,j}}^{(n)}}{2h}\right)^2+\left(\frac{f_{\mathrm{i,j+1}}^{(n)}-f_{\mathrm{i,j-1}}^{(n)}}{2h}\right)^2}};$$

$$(\Gamma_{11}^1)_{i,j}^{(n)}=\frac{G_{i,j}^{(n)}(E_{i+1,j}^{(n)}-E_{i-1,j}^{(n)})-2F_{i,j}^{(n)}(F_{i+1,j}^{(n)}-F_{i-1,j}^{(n)})+F_{i,j}^{(n)}(E_{i,j+1}^{(n)}-E_{i,j-1}^{(n)})}{4(E_{i,j}^{(n)}\cdot G_{i,j}^{(n)}-\left(F_{i,j}^{(n)}\right)^2)h}$$

$$(\Gamma_{11}^2)_{i,j}^{(n)}=\frac{2E_{i,j}^{(n)}(F_{i+1,j}^{(n)}-F_{i-1,j}^{(n)})-E_{i,j}^{(n)}(E_{i,j+1}^{(n)}-E_{i,j-1}^{(n)})-F_{i,j}^{(n)}(E_{i+1,j}^{(n)}-E_{i-1,j}^{(n)})}{4(E_{i,j}^{(n)}\cdot G_{i,j}^{(n)}-\left(F_{i,j}^{(n)}\right)^2)h}$$

$$(\Gamma_{22}^1)_{i,j}^{(n)}=\frac{2G_{i,j}^{(n)}(F_{i,j+1}^{(n)}-F_{i,j-1}^{(n)})-G_{i,j}^{(n)}(G_{i+1,j}^{(n)}-G_{i-1,j}^{(n)})-F_{i,j}^{(n)}(G_{i,j+1}^{(n)}-G_{i,j-1}^{(n)})}{4(E_{i,j}^{(n)}\cdot G_{i,j}^{(n)}-\left(F_{i,j}^{(n)}\right)^2)h}$$

$$(\Gamma_{22}^2)_{i,j}^{(n)}=\frac{E_{i,j}^{(n)}(G_{i,j+1}^{(n)}-G_{i,j-1}^{(n)})-2F_{i,j}^{(n)}(F_{i,j+1}^{(n)}-F_{i,j-1}^{(n)})+F_{i,j}^{(n)}(G_{i+1,j}^{(n)}-G_{i-1,j}^{(n)})}{4(E_{i,j}^{(n)}\cdot G_{i,j}^{(n)}-\left(F_{i,j}^{(n)}\right)^2)h}.$$

The matrix formulation of equation set (1) can be expressed as,

$$\begin{cases}\mathbf{A_1}\cdot\mathbf{z^{(n+1)}}=\mathbf{d_1^{(n)}}\\ \mathbf{A_2}\cdot\mathbf{z^{(n+1)}}=\mathbf{d_2^{(n)}}\end{cases}\tag{2}$$

where

$$\mathbf{z}^{(n+1)}=\left(f_{1,1}^{(n+1)},...,f_{1,J}^{(n+1)},.......,f_{I,1}^{(n+1)},...,f_{I,J}^{(n+1)}\right)^T=\left(z_1^{(n+1)},...,z_J^{(n+1)},...,z_{(I-1)\cdot J+1}^{(n+1)},...,z_{I\cdot J}^{(n+1)}\right)^T$$

, $z_{(i-1)\cdot J+j}^{(n+1)}=f_{i,j}^{(n+1)}$; $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively represent coefficient matrixes of the first equation and the second equation in equation set (1); $\mathbf{d}_1^{(n)}$ and $\mathbf{d}_2^{(n)}$ are respectively the right-hand vectors of equation set (1).

If the kth sampling point is located at the lattice $(x_i,y_j)$ in the computational domain, the simulation value should be equal to or approximate to the sampling value at this lattice, HASM-OC could be formulated as,

$$\begin{cases}\min\left\|\mathbf{A}\cdot\mathbf{z}^{(n+1)}-\mathbf{d}^{(n)}\right\|_2\\ s.t.\\ \qquad\mathbf{S}\cdot\mathbf{z}^{(n+1)}=\mathbf{k}\\ \qquad\mathbf{l}_b<\mathbf{z}^{(n+1)}<\mathbf{u}_b\end{cases}\tag{3}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$; $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1^{(n)} \\ \mathbf{d}_2^{(n)} \end{bmatrix}$; $\mathbf{S} \in R^{S_p \times (I \times J)}$ represents sampling matrix, $S_p$ is total number of sampling points; $S(p,(i-2)\cdot I + j - 1) = 1$; $\mathbf{k} \in R^{S_p \times 1}$ represents sampling vector and $K(p) = \bar{f}_{i,j}$; $lb_{(i-1)\cdot J+j}$ and $ub_{(i-1)\cdot J+j}$ are respectively lower and upper bounds of $f(x,y)$ at lattice $(x_i, y_j)$, in which $1 \le i \le I$ and $1 \le j \le J$; $\mathbf{l}_b < \mathbf{z}^{(n+1)} < \mathbf{u}_b$, $\mathbf{l}_b = (lb_1, \cdots, lb_J, \cdots, lb_{(I-1)\cdot J+1}, \cdots, lb_{I\cdot J})^T$, $\mathbf{u}_b = (ub_1, \cdots, ub_J, \cdots, ub_{(I-1)\cdot J+1}, \cdots, ub_{I\cdot J})^T$.

A contour tree is an abstraction of a scalar field that encodes the nesting relationships of isosurfaces (Carr *et al*., 2010). The contour tree (Chen *et al*., 2004) is constructed and used to determine the range of elevation at the central point of every lattice, which include 3 steps, 1) to sample the scattered points and contour lines passing through or almost passing through the central points of lattices to establish the equality constraints of elevations; 2) to get lower and upper bounds (contour elevation) of the unsampled lattices on the basis of the contour tree; 3) to transform mixed constraints into boundary constraints for simplifying simulation by a slight perturbation to the sampled elevations. In other words, HASM-OC includes equality constraints and boundary constraints. The step 3 make HASM-OC only deal with boundary constraints.

## 3. Validations of HASM-OC

It was believed that thin plate spline (TPS) provided accurate, operationally straightforward and computationally efficient solutions to the problem of the spatial interpolation (Hutchinson, 1995). TPS can be used to calculate regular grid digital terrain models (DTMs) from arbitrarily large topographic data sets. It automatically removes spurious sinks or pits by imposing a drainage enforcement condition on the fitted grid values and eliminates one of the main weaknesses of elevation grids produced by interpolation techniques for general purposes. TPS is a kind of optimal control method. In the meanwhile, TPS is a two-dimensional analog of the cubic spline in one dimension. TPS fits a spline function to the observations. The fitted function agrees with the observation values at the observation points.

The numerical tests and real-world studies (Yue *et al*., 2007, 2009; Yue and Song, 2008; Yue and Wang, 2010) demonstrated that accuracy of HASM was much higher than the classical methods such as inverse distance weight (IDW), triangulated irregular network (TIN), ordinary Kriging and cubic spline. In this paper, HASM-OC is validated by comparing with TPS.

### 3.1. Numerical test

Gaussian synthetic surface is selected as a numerical test surface. Its computational domain is $[-3,3]\times[-3,3]$. The test surface has the following formulation,

$$f(x,y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10(x/5 - x^3 - y^5)e^{-x^2-y^2} - e^{-(x+1)^2-y^2}/3 \quad (4)$$

where $-6.5510 \langle f(x,y) \langle 8.1062$.

The numerical test process include 5 steps: 1) to create a digital terrain model (DTM) with 2001×2001 grid cells by using the Gaussian synthetic surface, 2) to convert the DTM into contour lines that are regarded as the original contour lines, 3) to simulate the DTM using dataset of the original contour lines by means of both HASM-OC and TPS, 4) to convert the simulated DTM into contour lines that are termed retrieved contour lines, and 5) to compare errors created by HASM-OC and TPS, in which error is defined as difference between the original contour lines at step 2 and the retrieved contour lines at step 4.

At step 2, different contour intervals of 2 m, 1 m, 0.5 m, 0.2 m and 0.1 m are selected to analyze effects of densities of contour lines on simulation errors. At step 3, various spatial resolutions of 0.12 m×0.12 m, 0.06 m×0.06 m, 0.03 m×0.03 m, 0.015 m×0.015 m and 0.0075 m×0.0075 m are chosen to simulate the impact of different spatial resolutions on accuracy. 25 numerical tests are conducted totally. Input parameters of TPS are its default values of ArcGIS 9.0. At step 5, error indexes include contour-line-number (CLN) difference between the original contour lines and the retrieved contour lines, ratio of intersection area between the original contour lines and the retrieved contour lines to total length of the original contour lines (RIAL), and root mean square error (RMSE).

The contour interval, 2 m, is selected to display contour-line-number difference between the original contour lines and the retrieved contour lines because the difference is difficult to be visually compared when the contour lines are too dense. When contour interval is 0.1 m, two contour lines of the original ones could not be retrieved on spatial resolution of 0.12 m by HASM-OC. The reason is that the spatial resolutions are too coarse and there is no any central point of a lattice between the two contour lines. When the spatial resolution becomes finer enough, this problem is able to be solved (Table 1).

**Table 1:** CLN comparison between original contour lines and retrieved contour lines by HASM-OC

| **Spatial resolution \ contour interval** | **2** | **1** | **0.5** | **0.2** | **0.1** |
|---|---|---|---|---|---|
| **0.12** | 12 | 24 | 47 | 113 | 224 |
| **0.06** | 12 | 24 | 47 | 113 | 225 |
| **0.03** | 12 | 24 | 47 | 113 | 226 |
| **0.015** | 12 | 24 | 47 | 113 | 226 |
| **0.0075** | 12 | 24 | 47 | 113 | 226 |
| The number of the original contour lines | **12** | **24** | **47** | **113** | **226** |

However, the contour lines retrieved by TPS vary oscillationally with the spatial resolution becoming finer and the simulated CLN almost always is different from the original one (Table 2). An exception is that the retrieved CLN equals to the original one when spatial resolution is 0.06 m and contour intervals are 2 m and 0.5 m, which is a mere coincidence because TPS fails to retrieve one of original contour lines but add one false contour line. Simulation results of TPS are very unstable.

**Table 2:** CLN comparison between original contour lines and retrieved contour lines by TPS

| **Spatial resolution \ contour interval** | **2** | **1** | **0.5** | **0.2** | **0.1** |
|---|---|---|---|---|---|
| **0.12** | 11 | 21 | 42 | 107 | 215 |
| **0.06** | 12 | 23 | 47 | 119 | 239 |
| **0.03** | 15 | 27 | 52 | 129 | 249 |
| **0.015** | 14 | 27 | 56 | 131 | 261 |
| **0.0075** | 15 | 29 | 60 | 140 | 257 |
| The number of the original contour lines | **12** | **24** | **47** | **113** | **226** |

## 4. Conclusions and discussion

A new method for optimal control of HASM is developed to further improve accuracy of high accuracy surface modeling (HASM). We conduct a numerical test on synthetic Gaussian surface to assess its accuracy by comparing its simulation results with the ones of TPS because TPS is also one kind of optimal control method. The tests indicate that retrieved contour lines by HASM-OC match their corresponding original contour lines very well on appropriate spatial resolutions. HASM-OC gives a much better simulation results than TPS does in almost all cases. Even when TPS reports error in its simulation processes of several test regions, which means that the simulation is unable to continue, HASM-OC can successfully complete its simulation in these test regions.

The computational speed will be a major challenge faced by HASM-OC because of its complex formulation. We might be able to find a solution for this challenge by employing adaptive algorithm and parallel computing technology.

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