Tail probability of Extremes for Bivariate Matérn Field

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Abstract

Bivariate Matérn fields are widely used in the multivariate spatial model setting. They may represent two types of measurements obtained at spatial locations, such as the surface pressure and temperature. In this paper, we obtain an explicit form on the asymptotics of the probability that the maximum of each measurement in a fixed region exceeding a threshold simultaneously.

Keywords: Cross Dependence, Bivariate Matérn Field, Tail Asymptotics, Extremes, Double Sum Method

1. Introduction

Our work is motivated by the increasing need for analyzing multivariate measurements obtained at spatial locations (Gelfand et al., 2010; Wackernagel, 2003). A common example is modeling the environment data, which usually have several types of observations, such as the ozone level, PM2.5, precipitation, temperature and so on. There is a large literature on statistical models for one type of observation (Cressie, 1993; Stein, 1999). Yet, it's practically meaningful and theoretically more challenging to model multivariate measurements jointly. One of the reasons is that a weather event or a pollution level is usually affected by several factors, and these factors not only vary with locations, but also correlated with each other. So in order to describe the phenomenon better, it is important to consider the dependence structure both for the factor itself and among factors. The latter can be described by the cross-correlation structure (Kleiber and Genton, 2013; Apanasovich et al., 2012; Apanasovich and Genton, 2010).

Gneiting et al. (2010) introduced the full bivariate Matérn field $X(t) = (X_1(t), X_2(t))$, which is a $\mathbb{R}^2$-valued, stationary Gaussian random field on $\mathbb{R}^d$ with zero mean and matrix-valued Matérn covariance functions. As spatially correlated error field, this model was applied to probabilistic weather field forecasting for surface pressure and temperature over the North American Pacific Northwest. For the important role of the bivariate Matérn field in multivariate spatial modeling, we study the probability that the maximum of the two measurements exceeding a threshold within a bounded region D simultaneously (Anshin, 2006; Hashorva and Ji, 2014; Adler and Taylor, 2007; Piterbarg, 1996), i.e.,

$$\mathbb{P}\left(\max_{s \in D} X_1(s) > u, \max_{t \in D} X_2(t) > u \right), \quad \text{as } u \to \infty. \tag{1}$$

Applying the current theoretical work about the tail probability of extremes for bivariate Gaussian random field (Zhou and Xiao, 2014), we obtain an explicit form for the asymptotics of the tail probability (1).

2. Main results and Discussion

In this section, we specify the bivairate Matérn field in detail and state our main theorem with some discussions. First, we define the Matérn correlation function $M(h|\nu, a)$ with parameters $a > 0$ and $\nu > 0$ as follows.

$$M(h|\nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)}(a|h|)^{\nu} K_{\nu}(a|h|), \tag{2}$$

where K_ν is a modified Bessel function of the second kind with $\nu > 0$. Then, the matrix-valued covariance functions of the bivariate Matérn field is given by

$$C(h) = \begin{pmatrix} c_{11}(h) & c_{12}(h) \\ c_{21}(h) & c_{22}(h) \end{pmatrix}, \tag{3}$$

where $c_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} c_{11}(h) &= \sigma_1^2 M(h|\nu_1, a_1) \\ c_{22}(h) &= \sigma_2^2 M(h|\nu_2, a_2) \\ c_{12}(h) &= C_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}), \end{aligned} \tag{4}$$

with $a_1, a_2, a_{12}, \sigma_1, \sigma_2 > 0$ and $\rho \in (-1, 1)$.

Gneiting et al. (2010) proved that the above model is valid (i.e., the matrix $C(h)$ in (3) is non-negative definite) if and only if

$$\begin{aligned} \rho^2 \leq \frac{\Gamma(\nu_1 + d/2)}{\Gamma(\nu_1)} \frac{\Gamma(\nu_2 + d/2)}{\Gamma(\nu_2)} \frac{\Gamma(\nu_{12})^2}{\Gamma(\nu_{12} + d/2)^2} \frac{a_1^{2\nu_1} a_2^{2\nu_2}}{a_{12}^{4\nu_{12}}} \\ \times \inf_{t \geq 0} \frac{(a_{12}^2 + t^2)^{2\nu_{12}+d}}{(a_1^2 + t^2)^{\nu_1 + d/2}(a_2^2 + t^2)^{\nu_2 + d/2}}. \end{aligned} \tag{5}$$

Especially, when $a_1 = a_2 = a_{12} = 1$, the above condition is reduced to

$$\rho^2 \leq \frac{\Gamma(\nu_1 + d/2)}{\Gamma(\nu_1)} \frac{\Gamma(\nu_2 + d/2)}{\Gamma(\nu_2)} \frac{\Gamma(\nu_{12})^2}{\Gamma(\nu_{12} + d/2)^2}, \tag{6}$$

in which case the choice of ρ is fairly flexible.

Here we will focus on the standardized nonsmooth bivariate Matérn field, that is $\sigma_1 = \sigma_2 = 1$, $\nu_1, \nu_2 \in (0, 1)$. Without loss of generality, we assume $\nu_1 \leq \nu_2$. As a technical assumption, we assume that the smoothness parameter of the cross correlation $\nu_{12} > 1$. Under these conditions, we can apply Theorem 2.1 in Xiao (1995) to show that the sample function $t \mapsto X(t) \in \mathbb{R}^2$ generates various random fractals. In particular, let $X([0,1]^d) = \{X(t) : t \in [0,1]^d\}$ and $\mathrm{Gr}X([0,1]^d) = \{(t, X(t)) : t \in [0,1]^d\}$ be the range and graph of X on $[0,1]^d$, respectively, then

$$\dim_{\mathrm{H}} X([0,1]^d) = \min\left\{2, \frac{d}{\nu_1}, \frac{d + \nu_2 - \nu_1}{\nu_2}\right\}, \quad \text{a.s.}$$

and

$$\dim_{\mathrm{H}} \mathrm{Gr}X([0,1]^d) = \min\left\{\frac{d}{\nu_1}, \frac{d + \nu_2 - \nu_1}{\nu_2}, d + 2 - (\nu_1 + \nu_2)\right\}, \quad \text{a.s.}$$

In the above, $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

Before moving to the tail probability of extremes of the bivariate Matérn field, let's consider the tail probability of any pair of the two measurements

$\{(X_1(s), X_2(t)) \mid s, t \in D\}$. By the covariance structure of $X(\cdot)$ given above, we can see the measurements $(X_1(s), X_2(t))$ is a bivariate Gaussian random vector with correlation $c_{12}(t-s)$. Let

$$\Psi(u, \gamma) := \frac{(1+\gamma)^2}{2\pi u^2 \sqrt{1-\gamma^2}} \exp\left(-\frac{u^2}{1+\gamma}\right).$$

It is known that (Ladneva and Piterbarg, 2000)

$$\mathbb{P}(X_1(s) > u, X_2(t) > u) = \Psi(u, c_{12}(s-t))(1+o(1)), \text{ as } u \to \infty, \tag{7}$$

The exponential decay rate of the tail probability above is determined by the cross correlation $c_{12}(t-s)$. So we can expect the maximum cross correlation over the region D may dominate the exponential decay rate for the tail probability of extremes (1). Indeed, Piterbarg and Stamatovich (2005) have proven this statement. Specifically,

$$\begin{aligned} &\log \mathbb{P}\left(\max_{s\in D} X_1(s) > u, \max_{t\in D} X_2(t) > u\right) \\ &= -\frac{u^2}{1+\max_{s,t\in D} c_{12}(t-s)}(1+o(1)), \quad \text{as } u \to \infty. \end{aligned} \tag{8}$$

Further, let's investigate the property of the maximum cross correlation. When X_1 and X_2 are positively correlated (that is, $\rho > 0$ in (4)), $c_{12}(t-s)$ attains the maximum ρ at any pair of measurements located at $\{(s,s) \mid s \in D\}$. Yet, when they are negatively correlated (that is, $\rho < 0$), the maximum cross correlation is attained at pairs of measurements located on the boundary of D and farthest away from each other. To sum up, we have

$$\max_{s,t\in D} c_{12}(t-s) = \begin{cases} \rho, & \text{if } \rho > 0 \\ \rho M(r_D|\nu_{12}, a_{12}), & \text{if } \rho < 0 \end{cases}, \tag{9}$$

where $r_D := \max\{|s-t| \mid s,t \in D\}$.

Since the exponential decay rate varies a lot by the sign of the cross correlation, we are going to study the exact asymptotics of (1) in two cases separately. For simplicity, we will further suppose that $a_1 = a_2 = a_{12} = 1$.

Case 1: Positively correlated bivariate Matérn field Let $mes_d(\cdot)$ be the Lebesgue measure in $\mathbb{R}^d$ and H_ν be the Pickands constant with index ν [cf. Piterbarg (1996)]. By Zhou and Xiao (2014), we obtain our main theorem.

Theorem 1. *For the standardized nonsmooth bivariate Matérn field $X(t)$ with positive cross correlation, the tail probability of its extremes on the region D satisfies the following asymptotic property,*

$$\begin{aligned} &\mathbb{P}\left(\max_{s\in D} X_1(s) > u, \max_{t\in D} X_2(t) > u\right) \\ &= K \cdot mes_d(D) u^{d(\frac{1}{\nu_1}+\frac{1}{\nu_2}-1)} \Psi(u,\rho)(1+o(1)), \ \text{as } u \to \infty, \end{aligned} \tag{10}$$

where the constant

$$K = (2\pi)^{\frac{d}{2}} (-c''_{12}(0))^{-\frac{d}{2}} r_1^{\frac{d}{2\nu_1}} r_2^{\frac{d}{2\nu_2}} (1+\rho)^{-d(\frac{1}{\nu_1}+\frac{1}{\nu_2}-1)} H_{2\nu_1} H_{2\nu_2}$$

with $r_i = \frac{\Gamma(1-\nu_i)}{2^{2\nu_i}\Gamma(1+\nu_i)}, i = 1, 2.$

The above theorem demonstrates how the cross dependence structure and the smoothness of the surface affect the probability of extreme events to happen. When u is large, the maximum cross correlation ρ dominates the extreme probability, since the rate of exponential decay is $\frac{1}{1+\rho}$, which is consistent as the large deviation results in (8) and (9). Moreover, since $(X_1(\cdot), X_2(\cdot))$ attains maximum cross correlation on the domain $\{(s,s) \mid s \in D\}$, the extreme tail probability is proportional to the area of the region D, i.e., $mes_d(D)$. We refer to Zhou and Xiao (2014) for more detailed discussions.

The second important feature of (10) is that the smoothness parameters ν_1 and ν_2 of the surface X_1 and X_2, respectively, determine the polynomial power of the threshold u. As shown in Theorem 1, the smoother the surface is (that is, the larger ν_1, ν_2 are), the smaller the extreme tail probability is. When the surface is nonsmooth, the spatial correlation tends to be weak, which leads to more possibility to cause extreme events. We have conducted a simulation study to verify these phenomena. Figure 1 below are two realizations of the surface X_1 and X_2 on the region $[0,1]^2$ with $\nu_{12} = 1.1$ and $\rho = 0.35$ but varying smoothness parameters ν_1 and ν_2.

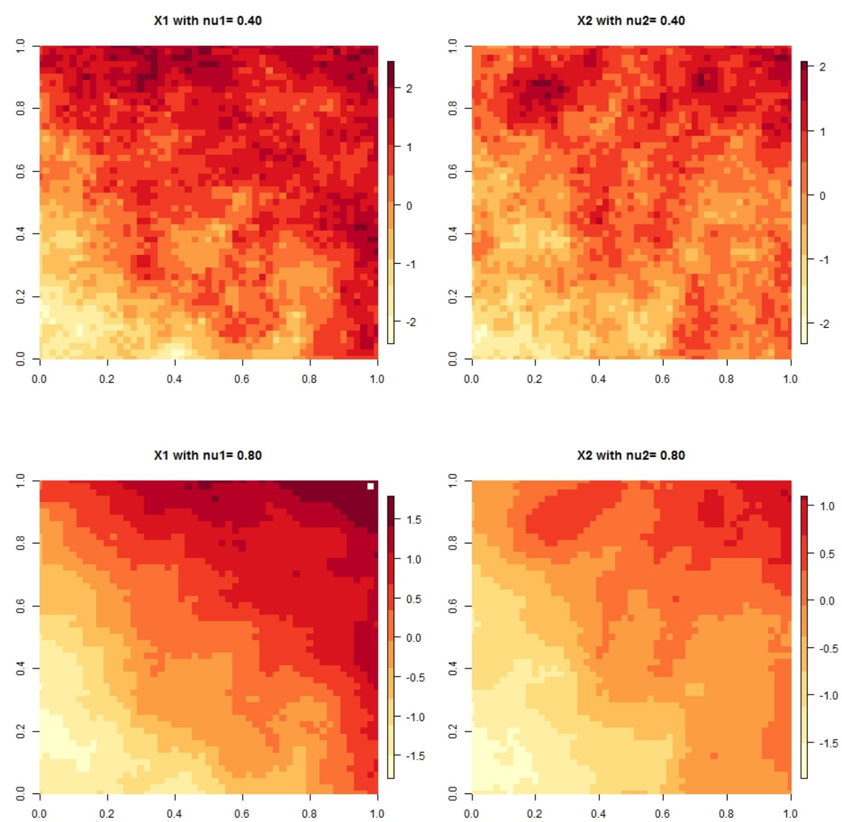

Figure 1: Top: Bivariate Matérn Field (X_1, X_2) with $\nu_1 = \nu_2 = .4$ and $\rho = 0.35$; Bottom: Bivariate Matérn Field (X_1, X_2) with $\nu_1 = \nu_2 = .8$ and $\rho = 0.35$

Case 2: Negatively correlated bivariate Matérn field By the proof of Theorem 1 (Zhou and Xiao, 2014), we found the region where the cross correlation attains maximum plays a key role to determine the tail probability (1). As mentioned before, the cross correlation of negatively correlated bivariate Matérn field attains its maximum when the measurements are observed on the boundary of D and are farthest away from each other. So the geometry of the boundary will affect the tail probability a lot. For example, if we choose $D = [0,1]^2$, the cross correlation attains maximum when two measurements are observed at the following four pairs of locations $\{((0,0),(1,1)),((0,1),(1,0)),((1,0),(0,1)),((1,1),(0,0))\}$, all of which are the vertices of the square $[0,1]^2$. The asymptotics of the tail probability (1) for this case is still under consideration.

3. Conclusion and future work

The tail probability of extremes for positively correlated bivariate Matérn field has been established explicitly in the paper. Our results show that the tail probability is dominated by the area where the cross correlation attains its maximum, which means only the observations inside or around those area contribute the most. It could be helpful if we are going to design an efficient algorithm to simulate the tail probability of extremes for multivariate random field. The basic results here motivate us to investigate the problem further in the future.

The cross correlation structure of bivariate Matérn field has some restrictions. First of all, being a valid bivariate Matérn model, the maximum cross correlation should satisfy (5). It may cause the choice of cross correlation ρ being limited. For example, if we choose $\nu_1 = \nu_2 = 0.5$, $\nu_{12} = 1.1$ and $d = 2$, the maximum cross correlation is no greater than 0.455 by (6). Yet, in Zhou and Xiao (2014), we've established the explicit form for the tail probability of extremes for a more general class of bivariate random field which can partially solve this issue. Second, under the framework of bivariate Matérn model, the cross correlation of two measurements attains the maximum at the same location. Yet, in some applications, there do exist "delay" effect such that this assumption fails (Wackernagel, 2003). If we consider the "delay" effect in the spatial domain, it will take some extra work to establish precise asymptotics for the tail probability of the double extremes.

The theorem we've obtained is only applied to a uniform threshold u. A natural interesting problem is to study the asymptotics of the probability that the extremes exceeding different level thresholds, i.e., given $\beta_1, \beta_2 > 0$,

$$\mathbb{P}\left(\max_{s\in D} X_1(s) > \beta_1 u, \max_{t\in D} X_2(t) > \beta_2 u\right), \text{ as } u \to \infty.$$

Solution of this problem requires different method and will be considered elsewhere.

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