Model testing for spatial strong mixing data

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Abstract

In analysing the distribution of a variable in a space, each value is subject not only to the source of the phenomenon but also to its localisation. In this paper, we fit the model of the distribution, taking explicitly into account the spatial autocorrelation among the observed data. To this end we first suppose that the observations are generated by a strong-mixing random field. Then, after estimating the density of the considered variable, we construct a test statistics in order to verify the goodness of fit of the observed spatial data. The proposed class of tests is a generalization of the classical chi-square-test and of the Neyman smooth test. The asymptotic behaviour of the test is analysed and some indications about its implementation are provided.

1. Introduction

Any discipline (geology, ecology, forestry, ...) that uses observations taken in different locations, or georeferenced, needs models that have been developed taking into account the dependence between the observations in the space. Before using inferential procedures or searching models, it needs often formulating and testing an hypothesis on the distribution of the considered variable. In environmental framework, for example, concentration of air pollutants or temperature are continuous variables occurring very often in wide statistical analysis. Kaiser et al. (2002) and other authors take the observations of PM10 at a given time and location to follow a lognormal distribution. On the other hand, a goodness of fit test can be used to check if a new simulation procedure for random field has produced the wanted distribution.

Usually the fit of the model to observations is tested by the classical χ^2 test (Rogers (1974), Ripley (1981), Cressie (1993)). But often it is not possible to assume the independence because usually the observations present spatial autocorrelation (Cliff and Ord (1973)) and it is more convenient to apply a test which takes into account this dependence.

Then we propose to generalize the classical χ^2 test, using the nonparametric density estimator by projection, to the case of correlated spatial data, analogously to the cases of time series data in Ignaccolo (2004) and of independent data in Bosq (2002) (see also Bosq's papers referenced there). This class of goodness of fit tests also contains the smooth test of Neyman (1937). To take explicitly into account the spatial autocorrelation among the observed data, we suppose that the observations are generated by a strong-mixing random field. In Section 2 we define the class of test, whose asymptotic behaviour is analysed in Section 3. Finally we give also some indications about the implementation of the test.

2. Definition of the test

Consider a strictly stationary random field $\mathbf{X} = (X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d)$, with $d \geq 1$, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space $(E, \mathcal{B})$. Assume that we observe $\mathbf{X}$ on a rectangular region $I_{\boldsymbol{n}} = \left\{\boldsymbol{i} : \boldsymbol{i} \in \mathbb{Z}^d, 1 \leq i_l \leq n_l, l = 1, \ldots, d\right\}$, we write $\boldsymbol{n} \to \infty$ if $\min_l n_l \to \infty$ and we set $n^* = |I_{\boldsymbol{n}}|$.
To consider the dependence between the observations we suppose that they are generated by a *strong-mixing* (or weak dependent) spatial process.
If $\Lambda \subset \mathbb{Z}^d$, let $\mathcal{A}_\Lambda$ be the σ-algebra generated by the X_ρ, $\rho \in \Lambda$. If $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$, let $d(\Lambda_1, \Lambda_2) = \inf\{d(\rho_1, \rho_2) : \rho_1 \in \Lambda_1, \rho_2 \in \Lambda_2\}$ and

$$\alpha(\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}) = \sup_{A_1 \in \mathcal{A}_{\Lambda_1}, A_2 \in \mathcal{A}_{\Lambda_2}} \left|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)\right|.$$

The random field $\mathbf{X}$ is said α-mixing (or strong mixing) if

$$\alpha_{uv}(n) = \sup\left\{\alpha(\mathcal{A}_{\Lambda_1}, \mathcal{A}_{\Lambda_2}) : d(\Lambda_1, \Lambda_2) \geq n, |\Lambda_1| \leq u, |\Lambda_2| \leq v\right\} \xrightarrow{n \to \infty} 0,$$

for any integers $u, v \geq 0$ (see Doukhan (1994)).
We want to test the hypothesis H_0 : "$X_{\boldsymbol{i}}$ has distribution μ" ($X_{\boldsymbol{i}} \sim \mu$), where μ is a probability measure on $(E, \mathcal{B})$. Let $\mathcal{P}$ be a family of probability measures on $(E, \mathcal{B})$ dominated by μ and let ν be the generic element of $\mathcal{P}$. We denote by f the probability density of ν with respect to μ, $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, and we assume that $f \in L^2(\mu)$, that is a separable Hilbert space with its scalar product $\langle f,\, g\rangle = \int fg\,\mathrm{d}\mu$.
For a fixed positive integer k let $\{e_0, e_1, \ldots, e_k\}$ be an orthonormal system with $e_0 \equiv 1$, in $L^2(\mu)$, which generates a subspace $E_k = \mathrm{span}\{e_0, e_1, \ldots, e_k\}$, with $\dim(E_k) = k + 1$.
The real valued function $K(x, t) = \sum_{j=0}^{k} e_j(x)e_j(t)$, with $(x, t) \in E \times E$ is the *reproducing kernel* of E_k (see Grenander (1981) and Fortet (1995)).
The density estimator of f by projection on E_k is defined as

$$f_{\boldsymbol{n}}(t) = \frac{1}{n^*} \sum_{\boldsymbol{i} \in I_{\boldsymbol{n}}} K(X_{\boldsymbol{i}}, t) = \sum_{j=0}^{k} \hat{a}_{j\boldsymbol{n}}\, e_j(t) \tag{1}$$

where $\hat{a}_{j\boldsymbol{n}} = \frac{1}{n^*} \sum_{\boldsymbol{i} \in I_{\boldsymbol{n}}} e_j(X_{\boldsymbol{i}})$ is the unbiased estimator of the Fourier coefficient $a_j = \langle f,\, e_j\rangle$.
Let us consider the $L^2(\mu)$–distance $d(f_{\boldsymbol{n}}, 1) = \|f_{\boldsymbol{n}} - 1\|$ between the estimated density and the true density $f_0 = \frac{\mathrm{d}\mu}{\mathrm{d}\mu} = 1$ under H_0, where $\|\cdot\|$ denotes the $L^2(\mu)$–norm.
Now we consider the statistic

$$T_{\boldsymbol{n}} = \sqrt{n^*}(f_{\boldsymbol{n}} - 1)$$

and its $L^2(\mu)$–norm $\|T_{\boldsymbol{n}}\| = \sqrt{n^*}\,\|f_{\boldsymbol{n}} - 1\|$ obtaining

$$\|T_{\boldsymbol{n}}\|^2 = n^* \left\|1 + \sum_{j=1}^{k} \hat{a}_{j\boldsymbol{n}} e_j(t) - 1\right\|^2 = n^* \sum_{j=1}^{k} \hat{a}_{j\boldsymbol{n}}^2.$$

We want to test $H_0 : X_{\boldsymbol{i}} \sim \mu$ versus $H_1 : X_{\boldsymbol{i}} \sim \nu \neq \mu$, that is $H_0 : f = 1$ versus $H_1 : f \neq 1$ considering the densities and also $H_0 :\ a_j = 0\ \forall j \geq 1$ versus $H_1 :\ \exists j \geq 1 : a_j \neq 0$ with respect to

the Fourier coefficients a_j.
Since the test is based on the deviation of the estimated density from the true density, it rejects H_0 for large values of $\|T_{\boldsymbol{n}}\|^2$. We shall prove in Section 3 that, under H_0, $\|T_{\boldsymbol{n}}\|^2$ converges in distribution to a linear combination of r.v.'s $U_j^2 \sim \chi_1^2$ where the coefficients λ_j^2 are the eigenvalues of the matrix Σ defined in (3).
So it is possible to carry out a test with rejection region $\left\{\|T_{\boldsymbol{n}}\|^2 > w\right\}$ and asymptotic size $\alpha \in]0,1[$ with w given by $\mathbb{P}\left(\sum_{j=1}^k \lambda_j^2 U_j^2 > w\right) = \alpha$, but the method requires the estimation of eigenvalues and the determination of quantiles.

Particular cases. Let $\{A_0, A_1, \ldots, A_k\}$ be a partition of E with $p_j = \mu(A_j) > 0$, $j = 0, \ldots, k$. For $e_j(\cdot) = p_j^{-1/2}\mathbb{I}_{A_j}(\cdot)$, the system $\{e_0, e_1, \ldots, e_k\}$ is orthonormal and it generates a subspace $E_k \subseteq L^2(\mu)$ that contains every constant. In this case the density estimator is the histogram and one easily obtains $\|T_{\boldsymbol{n}}\|^2 = \sum_{j=0}^k \frac{\left[\sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} \mathbb{I}_{A_j}(X_{\boldsymbol{i}}) - n^* p_j\right]^2}{n^* p_j}$ that is the test statistic used in the classical Pearson's χ^2 test.
Moreover, with an orthonormal system related to the Legendre polynomials the Neyman statistic (see also Rayner and Best (1989)) coincides with $\|T_{\boldsymbol{n}}\|^2$.

3. Large sample behaviour

The following notations are used throughout the paper. For each j we define the zero-mean real valued r.v.'s

$$Y_{\boldsymbol{i}j} = e_j(X_{\boldsymbol{i}}) - \mathbb{E}\left(e_j(X_{\boldsymbol{i}})\right), \quad \boldsymbol{i} \in I_{\boldsymbol{n}}$$

and we set

$$(n^*)^{-1/2} S_{nj} = (n^*)^{-1/2} \sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} Y_{\boldsymbol{i}j} = \sqrt{n^*}\left(\hat{a}_{j\boldsymbol{n}} - a_j\right)$$

with $n^* = |I_{\boldsymbol{n}}|$, noting that $a_j = \langle f, e_j\rangle = \mathbb{E}(e_j(X_{\boldsymbol{i}})) = \mathbb{E}(\hat{a}_{j\boldsymbol{n}})$.
Moreover we shall use

$$(n^*)^{-1/2}\boldsymbol{S}_n = \begin{pmatrix} (n^*)^{-1/2} S_{n1} \\ \vdots \\ (n^*)^{-1/2} S_{nk} \end{pmatrix} = \sqrt{n^*}\left[\begin{pmatrix} \hat{a}_{1n} \\ \vdots \\ \hat{a}_{kn} \end{pmatrix} - \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}\right] = \sqrt{n^*}\left(\boldsymbol{A}_n - \boldsymbol{a}\right)$$

and, for $\boldsymbol{i} \in I_{\boldsymbol{n}}$,

$$\boldsymbol{Y_i} = (Y_{\boldsymbol{i}1}, \ldots, Y_{\boldsymbol{i}k})^T$$

and the linear combination $V_{\boldsymbol{i}} = \sum_{j=1}^k c_j Y_{\boldsymbol{i}j} = \mathbf{c}^T \boldsymbol{Y_i}$ with $\boldsymbol{c} = (c_1, \ldots, c_k)^T \in \mathbb{R}^k$.
We consider the quadratic form

$$n^*(\boldsymbol{A}_n - \boldsymbol{a})^T I (\boldsymbol{A}_n - \boldsymbol{a}) = n^* \sum_{j=1}^k \left(\hat{a}_{j\boldsymbol{n}} - a_j\right)^2 := \left\|T'_{\boldsymbol{n}}\right\|^2$$

and we observe that under H_0 the statistic $\left\|T'_{\boldsymbol{n}}\right\|^2$ coincides with $\|T_{\boldsymbol{n}}\|^2$ because $a_j = 0$ for all j.
Consider a sequence $I_{\boldsymbol{n}}$ of finite subsets of $\mathbb{Z}^d$ that increases to $\mathbb{Z}^d$ and is such that $\lim_{\boldsymbol{n}\to\infty} \frac{|\partial I_{\boldsymbol{n}}|}{|I_{\boldsymbol{n}}|} = 0$. If

1. $\sum_{r=1}^\infty r^{d-1}\alpha_{uv}(r) < \infty$ for $u + v \le 4$ and $\alpha_{1,\infty}(r) = o(r^{-d})$;
2. for some $\delta > 0$ $\mathbb{E}(|V_{\boldsymbol{i}}|^{2+\delta}) < \infty$ and $\sum_{r=1}^\infty r^{d-1}\alpha_{1,1}(r)^{\delta/(2+\delta)} < \infty$;

3. $\sigma^2 > 0$ where $\sigma^2 = \sum_{\boldsymbol{i}\in\mathbb{Z}^d} \mathbb{E}(V_{\boldsymbol{0}} V_{\boldsymbol{i}})$;

then

$$\left\| T_{\boldsymbol{n}}^{'} \right\|^2 \xrightarrow{\mathrm{d}} \sum_{j=1}^{k} \lambda_j^2 U_j^2 \tag{2}$$

where the r.v.'s $U_j \sim \mathcal{N}(0,1)$ are independent and λ_j^2 are the eigenvalues of the matrix $\Sigma = (\sigma_{jl})_{j,l=1,\ldots,k}$ with

$$\sigma_{jl} = \sum_{\boldsymbol{v}\in\mathbb{Z}^d} \sigma_{jl}(\boldsymbol{v}) \tag{3}$$

where $\sigma_{jl}(\boldsymbol{v})$ is the crossed covariance function of the stationary bivariate spatial process $(Y_{\boldsymbol{t}j}, Y_{\boldsymbol{t}l})_{\boldsymbol{t}\in\mathbb{Z}^d}$. Since under H_0 the statistic $\left\| T_{\boldsymbol{n}}^{'} \right\|^2$ coincides with $\|T_{\boldsymbol{n}}\|^2$, the convergence (2) holds with particular values λ_j^{*2}.

To study the rate of convergence, we consider the distance

$$\Delta_n = \sup_{u\in\mathbb{R}} \left| \mathbb{P}\left(\left\| T_{\boldsymbol{n}}^{'} \right\|^2 \le u \right) - \mathbb{P}\left(\|U\|^2 \le u \right) \right| \tag{4}$$

with $\|U\|^2 = \sum_{j=1}^k \lambda_j^2 U_j^2$ (since $U = \sum_{j=1}^k \lambda_j U_j e_j$).

Then, if the mixing coefficients are exponentially decreasing, that is $\alpha_{uv}(n) = O\left(\mathrm{e}^{-an}\right)$ for $a > 0$, denoting $\sigma_n^2 = \mathsf{Var}\left(\sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} V_{\boldsymbol{i}}\right)$, the distance Δ_n satisfies

$$\Delta_n = O\left([\log \sigma_n]^{d[(1+\delta)\wedge 2]} \, \sigma_n^{-(\delta\wedge 1)} \right)$$

where $x \wedge y = \min(x,y)$, while if the mixing coefficients are arithmetically decreasing, that is $\alpha_{uv}(n) = O\left(n^{-a}\right)$ for $a > 0$, Δ_n satisfies

$$\Delta_n = O\left(\sigma_n^{-\xi}\right) \qquad \text{with} \qquad \xi = (\delta\wedge 1)\frac{2(b-1)}{2b + (\delta\wedge 1)} \qquad \text{and} \qquad b = \frac{a\delta(\delta\wedge 1)}{2d(2+\delta)\left[(1+\delta)\wedge 2\right]}.$$

Independent data. In the case of iid data in the sum in (3) remains only the term for $\boldsymbol{v} = \boldsymbol{0}$, so that $\sigma_{jl} = \mathbb{E}\left(Y_{\boldsymbol{0}j} Y_{\boldsymbol{0}l}\right)$. Then under the null hypothesis the element of Σ^* (that is Σ under H_0) becomes

$$\sigma_{jl}^* = \begin{cases} 0 & \text{for } j \ne l \\ 1 & \text{for } j = l \end{cases}$$

because $\sigma_{jj}^* = \mathbb{E}\left([e_j(X_{\boldsymbol{0}})]^2 \right) = \int e_j^2(x)\,\mathrm{d}\mu(x) = \|e_j\|^2 = 1$ and so $\Sigma^* = I_k$ where I_k is the identity matrix of order k. Then for all j one has $\lambda_j^{*2} = 1$ and the limit distribution is χ_k^2, as we have in the general case (see Bosq (2002)).

3.1 Proofs

We consider the linear combination $\boldsymbol{c}^T \boldsymbol{S}_n$ and with the previous notations we get

$$\boldsymbol{c}^T \boldsymbol{S}_n = \sum_{j=1}^k c_j S_{nj} = \sum_{j=1}^k c_j \sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} Y_{\boldsymbol{i}j} = \sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} \sum_{j=1}^k c_j Y_{\boldsymbol{i}j} = \sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} \boldsymbol{c}^T \boldsymbol{Y}_{\boldsymbol{i}} = \sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} V_{\boldsymbol{i}}.$$

With the previous assumptions, we can apply the Bolthausen CLT theorem (see Bolthausen (1982) or Rosenblatt (2000) p.56) to the zero-mean real valued random field $V_{\boldsymbol{i}}$ that is stationary and α-mixing (as we can write $V_{\boldsymbol{i}} = g(X_{\boldsymbol{i}})$ with g measurable and $\alpha_V(n) \le \alpha_X(n)$). Hence we obtain

$$\frac{\sum_{\boldsymbol{i}\in I_{\boldsymbol{n}}} V_{\boldsymbol{i}}}{\sigma\sqrt{n^*}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1)$$

that is

$$(n^*)^{-1/2}\boldsymbol{c}^T\boldsymbol{S}_n \xrightarrow{\text{d}} \mathcal{N}(0,\sigma^2).$$

If we can write $\sigma^2 = \boldsymbol{c}^T\Sigma\boldsymbol{c}$, with Σ symmetric and positive definite, we can apply the Cràmer–Wold device and we can obtain

$$\sqrt{n^*}(\boldsymbol{A}_n - \boldsymbol{a}) \xrightarrow{\text{d}} \boldsymbol{Z} \sim \mathcal{N}_k\left(\boldsymbol{0},\Sigma\right).$$

So for the quadratic forms we can have

$$n^*(\boldsymbol{A}_n - \boldsymbol{a})^T I(\boldsymbol{A}_n - \boldsymbol{a}) \xrightarrow{\text{d}} \boldsymbol{Z}^T I\boldsymbol{Z}$$

that is $\left\|T_{\boldsymbol{n}}'\right\|^2 \xrightarrow{\text{d}} \sum_{j=1}^k \lambda_j^2 U_j^2$ observing that $\boldsymbol{Z}^T I\boldsymbol{Z} = \sum_{j=1}^k \lambda_j^2 (U_j)^2 = \sum_{j=1}^k \lambda_j^2 \chi_1^2$.
So it remains to check that we can write $\sigma^2 = \boldsymbol{c}^T\Sigma\boldsymbol{c}$, identifying Σ.
From the Bolthausen CLT we have $\sigma^2 = \sum_{\boldsymbol{i}\in\mathbb{Z}^d}\mathbb{E}(V_{\boldsymbol{0}}V_{\boldsymbol{i}})$ and it holds

$$\mathbb{E}(V_{\boldsymbol{0}}V_{\boldsymbol{i}}) = \mathbb{E}\left(\left[\boldsymbol{c}^T\boldsymbol{Y}_{\boldsymbol{0}}\right]\left[\boldsymbol{c}^T\boldsymbol{Y}_{\boldsymbol{i}}\right]^T\right) = \mathbb{E}\left(\boldsymbol{c}^T\boldsymbol{Y}_{\boldsymbol{0}}\boldsymbol{Y}_{\boldsymbol{i}}^T\boldsymbol{c}\right) = \boldsymbol{c}^T\,\mathbb{E}\left(\boldsymbol{Y}_{\boldsymbol{0}}\boldsymbol{Y}_{\boldsymbol{i}}^T\right)\boldsymbol{c} = \boldsymbol{c}^T\Sigma_{\boldsymbol{0i}}\boldsymbol{c}$$

denoting $\Sigma_{\boldsymbol{0i}} = \mathbb{E}\left(\boldsymbol{Y}_{\boldsymbol{0}}\boldsymbol{Y}_{\boldsymbol{i}}^T\right)$. Then the generic element of Σ is $\sigma_{jl} = \sum_{\boldsymbol{i}\in\mathbb{Z}^d}\mathbb{E}\left(Y_{\boldsymbol{0}j}Y_{\boldsymbol{i}l}\right)$.
By stationarity of $\boldsymbol{Y}_{\boldsymbol{i}}$ it follows $\mathsf{Cov}(\boldsymbol{Y}_{\boldsymbol{0}},\boldsymbol{Y}_{\boldsymbol{v}}) = \mathsf{Cov}(\boldsymbol{Y}_{\boldsymbol{t}},\boldsymbol{Y}_{\boldsymbol{t}+\boldsymbol{v}})$ and for its each element $\mathbb{E}\left(Y_{\boldsymbol{0}j}Y_{\boldsymbol{v}l}\right) = \mathbb{E}\left(Y_{\boldsymbol{t}j}Y_{\boldsymbol{t}+\boldsymbol{v},l}\right)$ so we denote

$$\sigma_{jl}(\boldsymbol{v}) = \mathbb{E}\left(Y_{\boldsymbol{t}j}Y_{\boldsymbol{t}+\boldsymbol{v},l}\right)$$

the crossed covariance of the stationary bivariate spatial process $(Y_{\boldsymbol{t}j},Y_{\boldsymbol{t}l})_{\boldsymbol{t}\in\mathbb{Z}^d}$. Moreover the equality $\sigma_{jl}(-\boldsymbol{v}) = \mathbb{E}\left(Y_{\boldsymbol{t}j}Y_{\boldsymbol{t}-\boldsymbol{v},l}\right) = \mathbb{E}\left(Y_{\boldsymbol{t}-\boldsymbol{v},l}Y_{\boldsymbol{t}j}\right) = \sigma_{lj}(\boldsymbol{v})$ holds.
So we can write $\sigma_{jl} = \sum_{\boldsymbol{v}\in\mathbb{Z}^d}\sigma_{jl}(\boldsymbol{v})$ and we have $\sigma_{lj} = \sum_{\boldsymbol{v}\in\mathbb{Z}^d}\sigma_{lj}(\boldsymbol{v}) = \sum_{\boldsymbol{v}\in\mathbb{Z}^d}\sigma_{jl}(-\boldsymbol{v}) = \sum_{\boldsymbol{u}\in\mathbb{Z}^d}\sigma_{jl}(\boldsymbol{u}) = \sigma_{jl}$, so that Σ is symmetric.
With respect to the rate of convergence we apply Theorems 2 and 3 in Guyon and Richardson (1984) (see also Doukhan (1994) p. 49) to the random field $V_{\boldsymbol{i}}$ and, posing $N_\sigma \sim \mathcal{N}(0,\sigma^2)$, we observe that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left((n^*)^{-1/2}\boldsymbol{c}^T\boldsymbol{S}_n \le t\right) - \mathbb{P}\left(N_\sigma \le t\right)\right| \\
&\quad = \sup_{v\in\mathbb{R}}\left|\mathbb{P}\left(\left\|\boldsymbol{c}^T\left[\sqrt{n^*}(\boldsymbol{A}_n - \boldsymbol{a})\right]\right\|^2 \le v^2\right) - \mathbb{P}\left(\left\|\boldsymbol{c}^T\boldsymbol{Z}\right\|^2 \le v^2\right)\right| \\
&\quad = \sup_{v\in\mathbb{R}}\left|\mathbb{P}\left(n^*\left\|\boldsymbol{A}_n - \boldsymbol{a}\right\|^2 \le \frac{v^2}{\|\boldsymbol{c}\|^2}\right) - \mathbb{P}\left(\|\boldsymbol{Z}\|^2 \le \frac{v^2}{\|\boldsymbol{c}\|^2}\right)\right| \\
&\quad = \Delta_n
\end{aligned}$$

with the position $u = \frac{v^2}{\|\boldsymbol{c}\|^2}$ as defined in (4).

4. Implementation

The limit distribution for the test depends on the eigenvalues λ_j^2 of the unknown matrix Σ. So to estimate these eigenvalues we have to estimate the elements σ_{jl}.
Let $I_{\boldsymbol{nv}} = \{\boldsymbol{t},\boldsymbol{s}\in I_{\boldsymbol{n}} : \boldsymbol{t}-\boldsymbol{s} = \boldsymbol{v}\}$ be the set of pairs in $I_{\boldsymbol{n}}$ for which the difference is $\boldsymbol{v}$. We can define an estimator for the crossed covariance $\sigma_{jl}(\boldsymbol{v})$ as

$$\hat{\sigma}_{jl}(\boldsymbol{v}) = |I_{\boldsymbol{nv}}|^{-1}\sum_{I_{\boldsymbol{nv}}} Y_{\boldsymbol{t}j}Y_{\boldsymbol{s}l}$$

for $|I_{\boldsymbol{nv}}| \geq 1$ (if $|I_{\boldsymbol{nv}}| = 0$ no empirical estimate is possible because no data has been collected with locations $\boldsymbol{t}, \boldsymbol{s}$ such that $\boldsymbol{t} - \boldsymbol{s} = \boldsymbol{v}$).

Then we consider $\Lambda_{\boldsymbol{n}} = \{\boldsymbol{v} : \boldsymbol{v} = \boldsymbol{t} - \boldsymbol{s} \text{ for } \boldsymbol{t}, \boldsymbol{s} \in I_{\boldsymbol{n}}\}$ and we can estimate σ_{jl} by

$$\hat{\sigma}_{jl} = \sum_{\Lambda_{\boldsymbol{n}}} \hat{\sigma}_{jl}(\boldsymbol{v}).$$

With the estimated eigenvalues we have a linear combination of χ^2 r.v.'s with 1 degree of freedom. Exact significance points for selected values of k and of the coefficients were published by several authors. Moreover, different evaluations of these quantiles have been proposed, by approximation of series expansions or by numerical methods. For further details see Johnson et al. (1994) and Mathai and Provost (1992).

To run a test of the proposed class we have to choose an orthonormal system. This choice can be related to the considered distribution, using a set of orthonormal polynomials linked to it.

We suggest to use the Meixner class (see Lancaster (1975) and Rayner and Best (1989) p.140), for which an orthogonal system can be obtained from the recurrence relation $P_{n+1}(x) = (x + n\lambda)P_n(x) + n(-\sigma^2 + (n-1)\gamma)P_{n-1}(x)$ for $n = 0, 1, 2, \ldots$ where $P_{-1}(x) = 0$, $P_1(x) = 1$ and $\sigma^2 = \mathsf{Var}(X)$ with $X = Y - \mathbb{E}(Y)$. The constants λ and γ can be determined from $P_2(0)$ and $P_3(0)$ and to have an orthonormal system $P_n(x)$ must be normalized by dividing by $s_n = \sqrt{\mathbb{E}\left([P_n(x)]^2\right)}$.

The Normal and the Gamma distributions belong to the Meixner class and, in particular, for the gaussian case $\{P_n(x)\}$ is the set of Hermite polynomials. Moreover the Poisson, the Binomial and the Negative Binomial distributions belong to this class too.

We observe that the proposed class of tests is defined for continuous variables. But if we are interested in the models describing the placing of units in the space (like the Poisson, the Binomial and the Negative Binomial) and it is not possible to assume the independence between the observations, we suggest to apply the proposed test choosing the polynomials linked to the hypothesized distribution according to Meixner.

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